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Gravitational interactions of integrable models

E. Abdalla* and M.C.B. Abdalla†

CERN-TH
CH-1211 Geneva 23 - Switzerland

International Centre for Theoretical Physics - ICTP
34100 Trieste, Italy

Abstract

We couple non-linear σ-models to Liouville gravity, showing that integrability properties of symmetric space models still hold for the matter sector. Using similar arguments for the fermionic counterpart, namely Gross–Neveu-type models, we verify that such conclusions must also hold for them, as recently suggested.

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* Permanent address: Instituto de Física - USP, C.P. 20516, S. Paulo, Brazil.
† Permanent address: Instituto de Física Teórica - UNESP, R. Pamplona 145, 01405-900, S. Paulo, Brazil.

Non-linear σ-models defined on a symmetric space $M = G/H$ are integrable.[1] Moreover they are classically conformally invariant and do not interact with a gravitational field, which is readily seen to cancel as we substitute the metric, written in the conformal gauge, into the Lagrangian.

However, in the quantum theory several new features arise. The first of them is the mass generation as arising from the constraint due to quantum fluctuations, which generate a vacuum expectation value for the Lagrange multiplier. This fact leads to a non-conformally-invariant term, thus coupling the matter fields explicitly to the Liouville field. Moreover the trace anomaly in the computation of the determinant of the d'Alembert operator leads to a Liouville term which has also analogous contributions from the gravitational ghosts. A further issue is the fact that, globally, we cannot use the conformal gauge in a general Riemann surface, leading, in these cases, to extra moduli integrations. However, we stay, for the moment, in a base space with a trivial global topology, since the anomaly is generally connected with the Liouville field.

In order not to overload our formulae we restrain ourselves to the $O(N)$ non-linear σ-model, or else the $\mathbb{C}P^{N-1}$ model. However the results are trivially generalized to any symmetric space. These examples are leading, since already in the case of flat space both are integrable, with the diference that in the former the integrability condition stays valid in the quantum theory, since the gauge group $H = SO(N-1)$ is simple, while in the latter a quantum anomaly arises, spoiling integrability. This fact remains true in a general Grassmannian, where the gauge group is $H = S(U(N-p) \times U(p))$, or $H = S(O(N-p) \times O(p))$, where anomalies generated by the gauge fields $SU(p) \times U(1)$, or $SO(p)$ spoil the conservation laws. This is an important issue for string theory, where the relevant quotient space is $SO(32)/SO(8) \times SO(24)$, as we comment later on.

The partition function for the $O(N)$-model is given by the expression[1]

$$\mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}g^{\mu\nu} \mathrm{e}^{i\int \mathrm{d}^2x \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i} \times \mathcal{D}\alpha \mathrm{e}^{i\int \mathrm{d}^2x\sqrt{-g}\frac{\alpha(x)}{2\sqrt{N}}\left[\varphi_i^2 - \frac{N}{2f}\right]} \mathcal{D}[\text{ghosts}]\mathrm{e}^{iS_{\text{grav}}[\text{ghosts}]} \quad . \tag{1}$$

It is defined in terms of a Weyl-invariant action, a non-Weyl-invariant constraint (due to the field $\alpha(x)$), and a non-Weyl-invariant measure. The Weyl non-invariance of the measure has been studied in ref. [2]. There, it has been proved that the scalar fields measure transforms under Weyl transformation $g' = \mathrm{e}^\sigma g$ as

$$\prod_{i=1}^{N} \mathcal{D}_{\mathrm{e}^\sigma g}\varphi_i = \mathrm{e}^{-\frac{iN}{48\pi}S_L}\mathcal{D}_g\varphi_i \quad , \tag{2}$$

where S_L is the Liouville action.

Since the φ_i-fields build the N-plet appearing asymptotically, this is the only contribution to the Liouville action beside that of the ghosts, which gives the usual contribution -26. Therefore, writing the metric as a Liouville factor times a residual metric $\hat{g}$, we are left with

$$\mathcal{Z} = \int \mathcal{D}\varphi\mathcal{D}\sigma \mathrm{e}^{i\int \mathrm{d}^2\xi\left[\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i\right]}\mathcal{D}\alpha\mathrm{e}^{i\int \mathrm{d}^2x\mathrm{e}^{\gamma\sigma}\frac{\alpha(x)}{2\sqrt{N}}\left[\varphi_i^2 - \frac{N}{2f}\right]} \times \mathcal{D}[\text{ghosts}^{(0)}]\mathrm{e}^{iS^{(0)}_{\text{grav}}[\text{ghosts}]}\mathrm{e}^{i\frac{26-N}{24\pi}\int \mathrm{d}^2\xi\left[\frac{1}{2}\partial^\alpha\sigma\partial_\alpha\sigma - \mu\mathrm{e}^{\gamma\sigma} + Q\hat{R}\sigma\right]} \quad , \tag{3}$$

where γ, μ, Q are parameters that include possible quantum corrections arising from renormalization effects. Actually, we are mostly interested in the case where the background metric $\hat{g}^{\mu\nu}$ corresponds to Minkowski space $\eta^{\mu\nu}$. In the $\mathbb{C}P^{N-1}$ model we find

$$\begin{aligned}\mathcal{Z} = \int & \mathcal{D}\bar{z}\mathcal{D}z\mathcal{D}\sigma\mathcal{D}A_\mu\mathcal{D}\alpha\mathcal{D}[\text{ghosts}^{(0)}]\mathrm{e}^{iS^{(0)}_{\text{grav}}[\text{ghosts}]} \\ & \times \mathrm{e}^{i\int \mathrm{d}^2\xi\left[\eta^{\mu\nu}\overline{D^\mu z}D_\nu z + \mathrm{e}^{\gamma\sigma}\frac{\alpha(x)}{\sqrt{N}}\left[\bar{z}z - \frac{N}{2f}\right]\right]} \\ & \times \mathrm{e}^{i\frac{13-N}{12\pi}\int \mathrm{d}^2\xi\left[\frac{1}{2}\partial^\alpha\sigma\partial_\alpha\sigma - \mu\mathrm{e}^{\gamma\sigma} + Q\hat{R}\sigma\right]} \quad .\end{aligned} \tag{4}$$

Weak gravitational fields in eq. (3) are formally obtained for $|N-26| \to \infty$. The semiclassical gravity is actually obtained for $N-26 \to -\infty$. The constraint displays a gravitational interaction, which is trivial in the sense that it may be absorbed in the α-field measure, since we may define $\tilde{\alpha} = \mathrm{e}^{\gamma\sigma}\alpha$, and the Jacobian

$$J = \det \mathrm{e}^{\gamma\sigma} \tag{5}$$

corresponds to a renormalization of the interaction of the Liouville field with the background curvature, namely $\delta\mathcal{L} \simeq \hat{R}\sigma$. Thus we separate, at the Lagrangian level, three sectors, namely σ-model, Liouville and ghost sectors. The $O(N)$ σ-model sector is integrable, even before the α-field redefinition, because both equations

$$\partial^\mu j^{ij}_\mu = 0 \quad , \tag{6a}$$

$$\left[\partial_\mu + \frac{2f}{N}j_\mu, \partial_\nu + \frac{2f}{N}j_\nu\right] = 0 \quad , \tag{6b}$$

obeyed by the Noether current $j_{\mu_{ij}} = \varphi_i \overleftrightarrow{\partial}_\mu \varphi_j$ (or $j_{\mu_{ij}} = z_i \overleftrightarrow{D}_\mu \bar{z}_j$ for $\mathbb{C}P^{N-1}$), still hold true classically, independently of the redefinition implied by eq. (5). However, quantum mechanically, models such as $\mathbb{C}P^{N-1}$, which are defined on a symmetric space G/H, where H is not simple, have an anomaly in eq. (6b), which is thus spoiled by quantum effects. For the $O(N)$-model, $H = O(N-1)$ is simple and eq. (6b) holds in the quantum theory. For the $\mathbb{C}P^{N-1}$-model, the gauge group is $H = SU(N-1) \otimes U(1)$, not simple, and allows anomalous terms. Any symmetric space model G/H with a simple gauge group H displays, in the quantum theory, an infinite number of conservation laws.[16] If H is not simple the model is anomalous; only certain couplings with fermions (as e.g. supersymmetric) render integrability back in the quantum theory.[10] The models $O(8,8)/O(8) \times O(8)$ and $O(8,24)/O(8) \times O(24)$ are anomalous.[1,10] Due to cancellation of anomalies in the supersymmetric case,[1,10] use of the non-local conservation laws may then be effective.[11,17] The first reanalysis that has to be carried out in the quantum theory with gravitational fields, is the issue of the Wilson expansion of the currents with Liouville fields present; namely, in order to obtain an infinite number of conservation laws, the first of them,

$$Q^{(2)} = \int \mathrm{d}x\mathrm{d}y J_0(t,x)\epsilon(x-y)J_0(t,y) - Z\int \mathrm{d}x J_1(t,x) \tag{7}$$

is well defined and conserved by means of a suitable definition of the renormalization constant Z, which amounts to analysing the short-distance behaviour of the product of the currents

$$J_\mu(x+\epsilon)J_\nu(x) = C^\rho_{\mu\nu}(\epsilon)J_\rho(x) + D^{\rho\sigma}_{\mu\nu}(\epsilon)\partial_\rho J_\sigma(x) + E^{\rho_i}_{\mu\nu}(\epsilon)\mathcal{O}_{\rho_i}(x) \quad , \tag{8}$$

verifying that the last term does not really occur, since it spoils the conservation law.[10] As a matter of fact, the problem is similar to the one discussed in relation to the pure matter case,[16] as far as one dresses the fields with the gravitational background, as we discuss below. The consequence is that if the gauge group is simple, there are contributions spoiling the higher conservation laws. In the $\mathbb{C}P^{N-1}$ case the anomaly is given by

$$\frac{dQ^{(2)}}{dt} = \frac{1}{2\pi}\int \mathrm{d}x \mathrm{e}^{\beta\sigma} z\overline{z} F_{\mu\nu}\epsilon^{\mu\nu} \quad . \tag{9}$$

Once more, when the gauge group H is simple, no source of anomaly arises, since there is no candidate to be dressed. The constant β is fixed imposing that the conformal dimension of the integrand be[6] one).

The theory cannot be completely defined before its constraint structure is solved. In fact, as in the case of WZW gauge interactions, the constraints play a crucial role in the definition of asymptotic states; different sectors are decoupled at the Lagrangian level, but the first class constraints relate them by means of the definition of the physical states only coupling to the remaining sectors of the theory. Since they are first class, they imply a choice of the physical states of the theory. Such constraints may be obtained by coupling the theory to external gravitational fields, as proposed in ref. [3] in the gauged WZW G/H-coset construction. In other words, they coupled to external gauge fields, which turn out to disappear due to field redefinition, and the variation of the partition function with respect to such external fields are first-class constraints! Here we have the analogous construction coupling the matter fields to a classical gravitational field. For the Liouville action we have

$$\mathcal{L}_L = \frac{1}{4\pi}\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \sigma R(g) - \mu\mathrm{e}^\sigma\right) \quad , \tag{10}$$

which for $g^{\mu\nu} = \mathrm{e}^{\sigma'}\hat{g}^{\mu\nu}$, and adding the contribution $\mathcal{L}_L[\sigma', \hat{g}; \mu = 0]$ arising from the matter/ghost system with a definite choice of renormalization, leads to

$$\begin{aligned} \mathcal{L}^{\mathrm{tot}}_L &= \mathcal{L}_L[\sigma, \mathrm{e}^{\sigma'}\hat{g}^{\mu\nu}] + \mathcal{L}_L[\sigma', \hat{g}^{\mu\nu}] \\ &= \frac{1}{4\pi}\sqrt{-\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu(\sigma+\sigma')\partial_\nu(\sigma+\sigma') + (\sigma+\sigma')R(\hat{g}) - \mu\mathrm{e}^{\sigma+\sigma'}\right) \\ &\equiv \mathcal{L}_L[\sigma+\sigma', \hat{g}] \quad , \end{aligned} \tag{11}$$

which is the analogue of the Polyakov–Wiegmann identity.[4] Therefore the partition function does not depend on σ', which leads to a (first-class) constraint, corresponding to the Wheeler–de Wit equation.

First-class constraints are realized as equations defining the physical states. The dynamics, on the other hand, is obtained from the corresponding (factorized) Lagrangian,

which is equivalent to eq. (6b), implying, in turn, higher conservation laws, and a factorizable S-matrix. Moreover, they imply also a Yangian-type algebra as described in ref. [5], and consequently a half-affine algebra, by means of a Lie–Poisson action.

The Wheeler–de Wit equation corresponds to the vanishing Hamiltonian of the composite matter–Liouville–ghost system when acting in a physical state. If we work on a fixed background where the matter fields are constrained ($\vec{\varphi}^2 = 1$) we obtain, for the Wheeler–de Wit equation (see p. 44 of ref. [6]):

$$\left[\left(\frac{\partial}{\partial\sigma} + \frac{Q}{2}\right)^2 + \sum_{i=1}^{N} \frac{\partial^2}{\partial\varphi_i^2} + 2 - \left(\frac{Q}{2}\right)^2\right]\tilde{\psi} = 0 \quad . \tag{12}$$

For $\tilde{\psi} = \mathrm{e}^{-\frac{Q}{2}\sigma}\psi$, we obtain the $(N+1)$-dimensional Helmholtz equation on a cylinder of unit radius (we suppose $\frac{N}{2f} = 1$), that is

$$\left[\frac{\partial^2}{\partial\sigma^2} + \sum_{i=1}^{N} \frac{\partial^2}{\partial\varphi_i^2} + 2 - \left(\frac{Q}{2}\right)^2\right]\psi = 0 \quad . \tag{13}$$

With the ansatz

$$\psi = \mathrm{e}^{-\beta\sigma}\chi(\varphi_i) \quad , \tag{14}$$

we obtain

$$\left[\sum_{i=1}^{N} \partial_i^2 + 2 - \left(\frac{Q}{2}\right)^2 + \beta^2\right]\chi \equiv \left[\sum_{i=1}^{N} \partial_i^2 + m^2\right]\chi = 0 \quad , \tag{15}$$

on an $(N-1)$-dimensional sphere. This equation is solvable in terms of the Gegenbauer polynomials for[7]

$$m^2 = l(N + l - 2) \quad , \tag{16}$$

where l is an integer. For $l = 0$ the solution is a constant and corresponds to the matter vacuum. Using

$$Q^2 = \frac{25 - N}{3} \quad , \tag{17}$$

we obtain

$$\beta = \sqrt{\frac{19}{3} + (l - \frac{1}{3})N + l(l-2)} \quad , \tag{18}$$

which is real for $l \geq 1$. Although simple, the dressing involves also an oscillatory term for $N \geq 20$, if $l = 0$, or $N \geq 26$.

The case of fermionic interactions can be dealt with similarly. The analysis of the chiral Gross–Neveu model was performed in [10] (see also [18]). There is no candidate to the anomaly term either, obeying the usual symmetry requirements. The only difference is that in the case of the Gross–Neveu model, the bare fermion has to be further dressed by an explicit Liouville exponential, since it has a non-trivial conformal dimension, that is, we need to define ψ as

$$\psi_{\mathrm{bare}} = \mathrm{e}^{\frac{1}{4}\sigma}\psi \quad , \tag{19}$$

after which we also obtain, for the redefined matter Lagrangian:

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \frac{1}{2g}\phi^2 - \phi\bar{\psi}\psi \quad , \tag{20}$$

where ϕ was also dressed as $\phi_{\rm bare} = \mathrm{e}^{\frac{1}{2}\sigma}\phi$. This latter dressing is similar to the dressing of the Lagrange multiplier in the σ-model case. The equations for the Noether current, $j^{ij}_\mu = \bar{\psi}^j \gamma_\mu \psi^i$, namely conservation and pseudo-current divergence equations, lead to the integrability condition and higher conservation laws since the only influence of the ϕ-field is through its relation with the ψ field, obtained from the Ward identities, corresponding to the ϕ equations of motion. Thus, quantum integrability holds true in this case as well, confirming ref. [9], where the integrability of the Gross–Neveu model coupled to gravity has been recently conjectured, using completely different methods.

A final remark concerns the issue of the infinite Yangian symmetry of the theory. The higher symmetries are described in terms of the non-local conserved charges of the theory $Q^{(n)}$, studied by several authors.[1,5,8,12] When appropriately defined (adding combinations of charges of lower genus) we find an algebra of the type

$$\{Q_a^{(m)}, Q_b^{(n)}\} = \mathrm{tr}\,\tau^a\tau^b Q^{(n+m)} - \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \mathrm{tr}\,\left(\tau^a Q^{(i)} Q^{(j)} \tau^b Q^{(m+n-i-j-2)}\right) \quad . \tag{21}$$

The above algebra is of the Yangian type,[13] as shown in refs. [5,14]. Yangians correspond to quantum group symmetries of many integrable models.[15] The Poisson-algebraic structure dictated by the quantum non-local charges in the $O(N)$-invariant case is valid for the matter sector, implying a "half"-affine algebra structure, started out of the $O(N)$ generator $Q^{(0)}_{ij}$, which acts locally, that is on a given field A:

$$\delta^{(0)}_{ij} A = \{Q^{(0)}_{ij}, A\} \quad , \tag{22}$$

while for the first non-trivial higher action we follow ref. [8] and define

$$\delta^{(1)}_{ij} A = \{Q^{(1)}_{ij}, A\} + c\left(Q^{(0)}_{ia}\delta^{(0)}_{aj} - Q^{(0)}_{ja}\delta^{(a)}_{ai}\right) A \quad , \tag{23}$$

where c is a constant to be adjusted[5,8] and one obtains an algebra which, as claimed, is half of the affine structure. Due to the above algebraic structure, the issue of conservation of the first charges flows to the higher ones, rendering the previous discussion quite general. However, the symmetric transformations as generated by the non-local charges are obtained from a Lie–Poisson action,[15] and not by the familiar Hamiltonian action. For the first few charges, this is exemplified in eq. (23), and the symmetry transformation satisfies

$$[\delta^{(m)}_{ij}, \delta^{(n)}_{kl}] = \left(\delta \circ \delta^{(m+n)}\right)_{ij,kl} \quad , \tag{24}$$

with the obvious notations for the $O(N)$ Lie-algebra indices i, j, k, l. However, we cannot find a Hamiltonian generator $G^{(n)}$ that realizes the symmetry action $\delta^{(n)}\phi = \{H^{(n)}, \phi\}$.

Generalization for models on arbitrary symmetric spaces is immediate. If the gauge group H is simple there is no quantum anomaly and one can essentially proceed as in the $O(N)$ case. This happens to be analogous in the case of super $\mathbb{C}P^{N-1}$, where the anomaly cancels.[10] More complicated, purely bosonic models are, generally speaking, anomalous, as e.g. Grassmannian non-linear σ-models, where $G = SU(N)$ and $H = S(U(p) \otimes U(N-p))$.

Recently, there have been proposals[11] to explain the string theory content using the algebraic structure contained in the higher conservation laws and the previously mentioned half-affine Lie algebra structure. In view of the above results, the discussion should be pursued, in the quantum theory, only in cases where there is no anomaly. Therefore we are forced to go to the supersymmetric case, which is in fact the most interesting as well, and where the anomaly generally cancels.[10]

Finally, we have to mention that in the quantum theory there are also field configurations not obeying the constraint $\vec{\varphi}^2 = 1$. Thus quantum states smear off such a background, leading to a perturbation of the form $\delta H = \frac{1}{2\sqrt{N}}\tilde{\alpha}(x)(\vec{\varphi}^2 - 1)$. For asymptotic fields arising from the integrable $O(N)$ model, it is reasonable to assume that this corresponds to a scalar field mass term, namely we substitute the field $\tilde{\alpha}(x)$ for its vacuum expectation value, which renders the resulting Wheeler–de Wit equation still separable and solvable.

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